Comment on 'Vortex structure of quantum eigenstates and classical periodic orbits in two-dimensional harmonic oscillators'

2005 J. Phys. A: Math. Gen. 38 2299

(http://iopscience.iop.org/0305-4470/38/10/N01)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 2299–2302 doi:10.1088/0305-4470/38/10/N01

COMMENT

Comment on 'Vortex structure of quantum eigenstates and classical periodic orbits in two-dimensional harmonic oscillators'

Adam J Makowski

Instytut Fizyki, Uniwersytet Mikołaja Kopernika, ul.Grudziądzka 5, 87-100 Toruń, Poland

E-mail: amak@phys.uni.torun.pl

Received 28 June 2004, in final form 11 January 2005
Published 23 February 2005
Online at stacks.iop.org/JPhysA/38/2299

Abstract

A simple correction to the results of the work by Chen Y F and Huang K F (2003 *J. Phys. A: Math. Gen.* **36** 7751) is proposed and its consequences are briefly discussed.

PACS numbers: 03.65.Vf, 03.65.Ge, 03.65.Fd

(Some figures in this article are in colour only in the electronic version)

In the recent paper [1] to this journal, Chen and Huang studied, among other things, a connection between classical Lissajous figures and quantum probability distributions concentrated on the figures. To this end, they used the $SU(2)$ coherent state for two non-interacting harmonic oscillators

$$\Psi_N^{p,q}(x, y, X, Y, \tau) = \frac{1}{(1+|\tau|^2)^{N/2}} \sum_{K=0}^{N} \binom{N}{K}^{1/2} \tau^K \Phi_{pK,q(N-K)}(x, y, X, Y) \tag{1}$$

where $\omega_x = \omega q$, $\omega_y = \omega p$, $X = \sqrt{2\hbar/(m_x\omega_x)}$, $Y = \sqrt{2\hbar/(m_y\omega_y)}$, τ is a complex, continuous parameter, $\tau = A\exp(\mathrm{i}\phi)$ and $\Phi_{pK,q(N-K)}$ is a product of the well-known eigensolutions for one-dimensional harmonic oscillators in x and y variables. The value of N plays here the role of the principal quantum number for the whole system.

Authors of the work [1] claim they have guessed the exact quantum–classical connection for coprime p and q values in the form

$$x(t) = \sqrt{2\langle x^2\rangle}\cos(q\omega t - \phi/p), \qquad y(t) = \sqrt{2\langle y^2\rangle}\cos(p\omega t) \tag{2}$$

where the expectation values of x^2 and y^2 were calculated in the state (1) and are

$$\langle x^2\rangle = \left(\frac{A^2}{1+A^2}pN + \frac{1}{2}\right)\frac{X^2}{2}, \qquad \langle y^2\rangle = \left(\frac{1}{1+A^2}qN + \frac{1}{2}\right)\frac{Y^2}{2}. \tag{3}$$

Chen and Huang wrote [1] '*...it should be remarked that the parameters in equation (2), which gives a connection between the wave pattern of a coherent state and classical Lissajous figures, are deduced from numerical calculations. It is difficult for us to give an analytical derivation for the expression of equation (2) at present*'.

We shall show in this comment that the Lissajous figures obtained from equations (2) do not cover the apogees of the quantum probability distribution $\left|\Psi_N^{p,q}(x, y, X, Y, \tau)\right|^2$, in contrast to what is said in [1]. We are able to prove the statement formally for the case of $p : q = 1 : 1$. For other relatively prime values of p and q we are able to do that only numerically.

Now, for simplicity, we take $X^2 = Y^2 = 2$. Then, for $p = q = 1$, the sum over K in equation (1) can be performed exactly [2], and we get

$$\Psi_N^{1,1}(x, y, \tau) = (2^N \pi N!)^{-1/2} \left(\frac{1+\tau^2}{1+|\tau|^2}\right)^{N/2} H_N\left(\frac{y + x\tau}{\sqrt{1+\tau^2}}\right) \exp[-(x^2+y^2)/2]. \tag{4}$$

Furthermore, for the choice $\tau = \pm\mathrm{i}\,(A = 1, \phi = \pm\pi/2)$, we can find from the properties of Hermite's polynomials that

$$\left[(1+\tau^2)^{N/2} H_N\left(\frac{y + x\tau}{\sqrt{1+\tau^2}}\right)\right]_{\tau=\pm\mathrm{i}} = 2^N (y \pm \mathrm{i}x)^N. \tag{5}$$

With this formula in mind, we have

$$\left|\Psi_N^{1,1}(x, y, \tau = \pm\mathrm{i})\right|^2 = \frac{(x^2+y^2)^N}{\pi N!} \exp[-(x^2+y^2)] \tag{6}$$

and, it is easy to prove that the maximum of the distribution lies on the circle $x^2 + y^2 = N$. For the maximum, we have $\left|\Psi_N^{1,1}\right|^2 = N^N \mathrm{e}^{-N}/(\pi N!)$ which, after using the Stirling's formula $N! \sim N^N \mathrm{e}^{-N}\sqrt{2\pi N}$, decays as $\left|\Psi_N^{1,1}\right|^2 \sim (\pi\sqrt{2\pi N})^{-1}$. Meanwhile, from the Chen and Huang's guessed equations (2), we will get $x^2 + y^2 = N + 1$. Though the difference first of all is well pronounced for low values of N, and practically plays no role for very high values, we may conclude that equations (2) do not give any exact connection between the classical figure (equations (2)) and the quantum state (equation (1)).

The corrected correspondence equations are supposed to be

$$x(t) = \sqrt{2\langle x^2\rangle - \hbar/(m_x\omega_x)}\cos(q\omega t - \phi/p), \qquad y(t) = \sqrt{2\langle y^2\rangle - \hbar/(m_y\omega_y)}\cos(p\omega t). \tag{7}$$

When $p : q = 1 : 1$ and $\tau = \pm\mathrm{i}$, we will obviously get for $\hbar/(m_x\omega_x) = \hbar/(m_y\omega_y) = 1$ a circle of radius $\sqrt{N}$, as it should be. For other combinations of coprime p and q, more complicated Lissajous figures appear, and the advantage of equations (7) over (2) can only be observed in detailed numerical tests. It follows from our tests that the curves calculated from equations (7) run along the apogees of the distribution $\left|\Psi_N^{p,q}(x, y, X, Y, \tau)\right|^2$, and those calculated from equations (2), are moved slightly away of them. With growing values of N, the difference becomes very small and in the Bohr's limit ($N \gg 1$), tends to zero.

A numerical example illustrating this point is given in figures 1 and 2 for $N = 2$ and $N = 10$, respectively. The Chen–Huang Lissajous trajectory of equations (2) (dotted) is compared with the proposed here corrected one from equations (7) (full line). As we can see, in each case, the latter curve runs exactly along the highest values of the quantum probability density whereas the former one is displaced beyond the values. The effect of our correction vanishes for higher values of the quantum number N. The tendency is visible in figure 2.

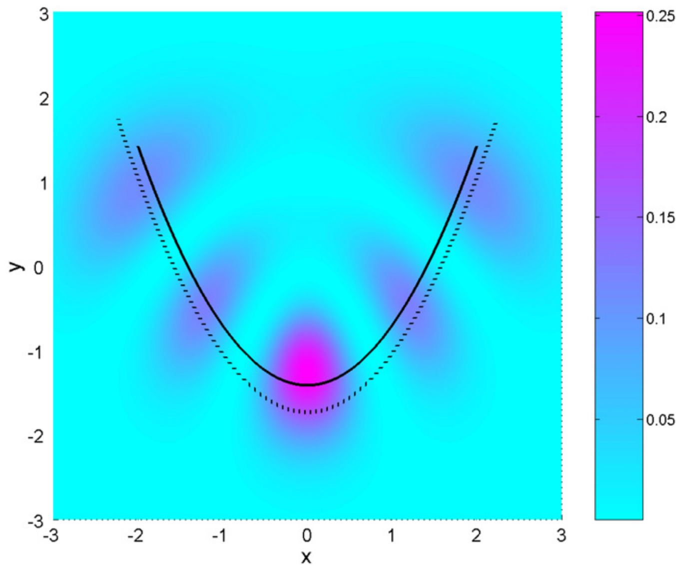

Figure 1. A comparison between $|\Psi_N^{p,q}(x, y, X, Y, \tau)|^2$ and the classical Lissajous figures from equations (2) (dotted) and equations (7) (full line) for $p = 2, q = 1$ with $\tau = 1(A = 1, \phi = 0), \hbar/(m_x\omega_x) = \hbar/(m_y\omega_y) = 1$ and $N = 2$.

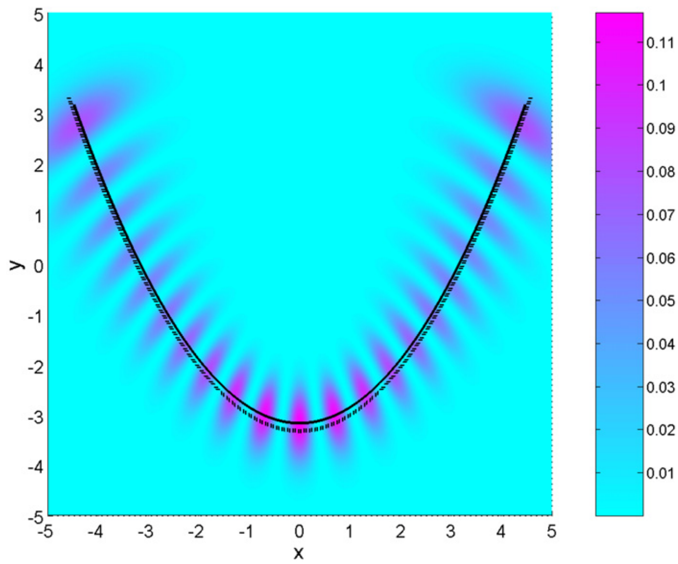

Figure 2. As in figure 1 but for $N = 10$.

Generally, finding *formally exact* close relations between the classical Lissajous figures and their corresponding quantum probability distributions remain still as an unsolved problem. It is very important for the quantum–classical correspondence since the formation of the Lissajous wave patterns in microchip laser resonators is at present a subject of intensive experimental studies [3–5].

References

[1] Chen Y F and Huang K F 2003 *J. Phys. A: Math. Gen.* **36** 7751

[2] Bateman H and Erdelyi A 1953 *Higher Transcendental Functions* vol 2 (New York: Mc Graw-Hill) chapter 10.13

[3] Chen Y F, Lan Y P and Huang K F 2003 *Phys. Rev.* A **68** 043803

[4] Chen Y F and Lan Y P 2002 *Phys. Rev.* A **66** 053812

[5] Chen Y F, Huang K F and Lan Y P 2003 *Opt. Lett.* **28** 1811